GENERALIZED B-STRONGLY B*-SEPARATION AXIOMS IN TOPOLOGICAL SPACES
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Abstract
The purpose of this paper is to introduce a new class of spaces via gbsb*-open sets and gbsb*-difference sets. Further we give some basic properties and their various characterizations.

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Introduction
D.Andrijevic[1] introduced the concept of b-open sets and characterized its topological properties. Caldas and Jafari [2], introduced and studied b-T0, b-T1, b-T2, b-D0, b-D1 and b-D2 via b-open sets after that Keskin and Noiri [5], introduced the notion of b-T1/2. A.Poongothai and P.Parimelazhagan,[6] introduced sb*-closed sets and we extend this concept into gbsb*-open sets[7].

In this chapter, we introduce a new classes of spaces called gbsb*-Tk spaces, for k=0, 1, 2, 1/2, gbsb*-Dk spaces, for k=0,1,2 and gbsb*-spaces. Also we study some basic properties and their various characterizations.

Preliminaries
Throughout this paper (X, τ) represents a topological space on which no separation axiom is assumed unless otherwise mentioned. (X, τ) will be replaced by X if there is no changes of confusion. For a subset A of a topological space X, cl(A) and int(A) denote the closure of A and the interior of A respectively. We recall the following definitions and results.

Definition 2.1.[1] Let (X, τ) be a topological space. A subset A of the space X is said to be b-open if A⊆int(cl(A))∪cl(int(A)) and b-closed if int(cl(A))∪cl(int(A)) ⊆ A.

Definition 2.2. Let (X, τ) be a topological space and A ⊆ X. The b-closure of A, denoted by bcl(A) and is defined by the intersection of all b-closed sets containing A.

Definition 2.3.[6] Let (X, τ) be a topological space and A ⊆ X. The b-closure of A, denoted by bcl(A) and is defined by the intersection of all b-closed sets containing A.

Definition 2.4.[7] A subset A of a topological space (X, τ) is called a generalized b-strongly b*-closed set (briefly, gbsb*-closed) if bcl(A) ⊆ U whenever A ⊆ U and U is b-open in (X, τ).

Definition 2.5.[7] The complement of the gbsb*-closed set is a gbsb*-open set. The collection of all gbsb*-open sets of X is denoted by gbsb*-O(X, τ).

Definition 2.6.[8] Let A be a subset of a topological space (X, τ). Then the union of all gbsb*-open sets contained in A is called the gbsb*-interior of A and it is denoted by gbsb*-int(A). That is gbsb*-int(A)=U{V:V⊆A and V∈gbsb*-O(X)}.
Definition 2.7. Let A be a subset of a topological space \((X, \tau)\). Then the intersection of all gbsb*-closed sets in \(X\) containing A is called the gbsb*-closure of A and it is denoted by gbsb*cl(A). That is gbsb*cl(A)\(=\cap\{F: A \subseteq F \text{ and } F \in \text{gbsb}*-\text{C}(X, \tau)\}\).

Theorem 2.8. Let A be a subset of a topological space \((X, \tau)\). Then
(i) A is gbsb*-open if and only if gbsb*int(A)=A.
(ii) A is gbsb*-closed if and only if gbsb*cl(A)=A.

Theorem 2.9. For every element \(x\) in a space \(X\), \(X\{x\}\) is either gbsb*-closed or sb*-open.

Theorem 2.10. Every closed set is sb*-closed.

Definition 2.11. Let \(X\) be a topological space and let \(x\in X\). A subset \(N\) of \(X\) is said to be a gbsb*-neighbourhood (shortly, gbsb*-nbhd) of \(x\) if there exists a gbsb*-open set \(U\) such that \(x\in U\subseteq N\).

Generalized b-strongly b*-T\(_k\) spaces

Definition 3.1. A topological space \((X, \tau)\) is said to be
(i) gbsb*-T\(_0\) if for each pair of distinct points \(x, y\) in \(X\), there exists a gbsb*-open set \(U\) in \(X\) such that either \(x\in U\) and \(y\not\in U\) or \(y\in U\) and \(x\not\in U\).
(ii) gbsb*-T\(_1\) if for each pair of distinct points \(x, y\) in \(X\), there exist two gbsb*-open sets \(U\) and \(V\) in \(X\) such that \(x\in U\) but \(y\not\in U\) and \(x\neq V\) and \(y\in V\).
(iii) gbsb*-T\(_2\) if for each pair of distinct points \(x, y\) in \(X\), there exist two disjoint gbsb*-open sets \(U\) and \(V\) in \(X\) such that \(x\in U\) and \(y\not\in U\) and \(x\neq V\) and \(y\in V\).
(iv) gbsb*-T\(_{1/2}\) if for every gbsb*-closed set is sb*-closed.
(v) gbsb*-space if every gbsb*-open set is open.

Theorem 3.2. A topological space \((X, \tau)\) is gbsb*-T\(_0\) if and only if for each pair of distinct points \(x, y\) in \(X\), gbsb*cl\(\{x\}\)\(\neq\)gbsb*cl\(\{y\}\).

Proof: Necessity: Suppose \(X\) is gbsb*-T\(_0\) and \(x,y\) are any two distinct points of \(X\). Then there exists a gbsb*-open set \(U\) containing \(x\) or \(y\), say \(x\) but not \(y\). Since \(U\) is gbsb*-open, \(X\cup U\) is a gbsb*-closed set which does not contain \(x\) but contains \(y\). Since gbsb*cl\(\{y\}\) is the smallest gbsb*-closed set containing \(y\), gbsb*cl\(\{y\}\)\(\subseteq X\cup U\). Then \(x\notin gbsb*cl\{y\}\). Hence gbsb*cl\(\{x\}\)\(\neq\)gbsb*cl\(\{y\}\).

Sufficiency: Suppose that \(x, y\in X\) with \(x\neq y\) and gbsb*cl\(\{x\}\)\(\neq\)gbsb*cl\(\{y\}\). Then there exists a point \(z\in X\) such that \(z\notin gbsb*cl\{x\}\) but \(z\notin gbsb*cl\{y\}\). Now, we claim that \(x\notin gbsb*cl\{y\}\). If \(x\notin gbsb*cl\{y\}\), then gbsb*cl\(\{x\}\)\(\subseteq\)gbsb*cl\(\{y\}\). This implies, \(z\notin gbsb*cl\{y\}\), which contradicts \(z\in gbsb*cl\{y\}\). Therefore \(x\notin gbsb*cl\{y\}\). Since gbsb*cl\(\{y\}\) is gbsb*-closed set containing \(y\) but not \(x\), then \(X\backslash gbsb*cl\{y\}\) is a gbsb*-open set containing \(x\) but not \(y\). Hence \(X\) is a gbsb*-T\(_0\) space.

Theorem 3.3. A topological space \((X, \tau)\) is gbsb*-T\(_1\) if and only if the singletons are gbsb*-closed sets.

Proof: Let \((X, \tau)\) be a gbsb*-T\(_1\) space and \(x\) be any point of \(X\). Let \(y\in X\backslash\{x\}\). Then \(y\neq x\) and so there exists a gbsb*-open set \(U\) containing \(y\) but not \(x\). That is \(y\in U\subseteq X\backslash\{x\}\). This implies, \(X\backslash\{x\}=U\cup\{y\}\subseteq X\backslash\{x\}\). Since the union of gbsb*-open sets is gbsb*-open, then \(X\backslash\{x\}\) is gbsb*-open containing \(y\) but not \(x\). Hence \(x\) is gbsb*-closed in \(X\). Conversely, suppose \(\{p\}\) is gbsb*-closed, for every \(p\in X\). Let \(x,y\in X\) with \(x\neq y\). Then \(y\notin X\backslash\{x\}\) and \(x\in X\backslash\{y\}\). Since \(\{x\}\) and \(\{y\}\) are gbsb*-closed sets in \(X\), then \(X\backslash\{x\}\) and \(X\backslash\{y\}\) are gbsb*-open sets in \(X\). Thus, we have a gbsb*-open set containing \(x\) but not \(y\) and a gbsb*-open set containing \(y\) but not \(x\). Hence \(X\) is a gbsb*-T\(_1\) space.
Theorem 3.4. A topological space \((X, \tau)\) is gbsb*-T\(_{1/2}\) if each singleton \(\{x\}\) of \(X\) is either sb*-closed or sb*-open.

Proof: Let \((X, \tau)\) be a gbsb*-T\(_{1/2}\) space.

Case (i): Suppose \(\{x\}\) is not sb*-closed. Then \(X\setminus\{x\}\) is not sb*-open. By Theorem 2.9, \(X\setminus\{x\}\) is gbsb*-closed. Since \(X\) is a gbsb*-T\(_{1/2}\) space, then \(X\setminus\{x\}\) is sb*-closed and hence \(\{x\}\) is sb*-open.

Case (ii): Suppose \(\{x\}\) is not sb*-closed. Then \(X\setminus\{x\}\) is not sb*-closed. By Theorem 2.9, \(X\setminus\{x\}\) is gbsb*-open in \(X\). Since \(X\) is a gbsb*-T\(_{1/2}\) space, then \(X\setminus\{x\}\) is sb*-open and hence \(\{x\}\) is sb*-closed.

Theorem 3.5. The following statements are equivalent for a topological space \(X\).

(i) \(X\) is gbsb*-T\(_2\).

(ii) For each \(x\in X\) and \(y\neq x\), there exists a gbsb*-open set \(U\) containing \(x\) such that \(y\notin \text{gbsb*cl}(U)\).

(iii) For each \(x\in X\), \(\cap\{\text{gbsb*cl}(U)/U\in \text{gbsb*O}(X, \tau)\text{ and }x\notin U\}\)=\(\{x\}\).

Proof:

(i)\(\Rightarrow\)(ii): Suppose \(X\) is gbsb*-T\(_2\). Then for \(x,y\in X\) with \(x\neq y\). Then there exists disjoint gbsb*-open sets \(U\) and \(V\) containing \(x\) and \(y\) respectively. Since \(V\) is gbsb*-open, then \(X\setminus V\) is gbsb*-closed containing \(U\). Hence gbsb*cl\((U)\subseteq X\setminus V\). Since \(y\notin V\), then \(y\notin X\setminus V\) and hence \(y\notin \text{gbsb*cl}(U)\).

(ii)\(\Rightarrow\)(iii): If there exists an element \(y\neq x\) in \(X\) such that \(y\notin \cap\{\text{gbsb*cl}(U)/U\in \text{gbsb*O}(X, \tau)\text{ and }x\notin U\}\), then \(y\notin \text{gbsb*cl}(U)\) for every gbsb*-open set \(U\) containing \(x\). This contradicts our assumption. So there exists no such an element \(y\). This proves (iii).

(iii)\(\Rightarrow\)(i): Let \(x,y\in X\) with \(x\neq y\). Then by our assumption, there exists a gbsb*-open set \(U\) containing \(x\) such that \(y\notin \text{gbsb*cl}(U)\). Let \(V=X\setminus \text{gbsb*cl}(U)\). Then \(V\) is gbsb*-open set containing \(y\). Also \(x\notin U\) and \(U\cap V=\emptyset\). Thus we have a disjoint gbsb*-open sets \(U\) and \(V\) containing \(x\) and \(y\) respectively. Hence \(X\) is a gbsb*-T\(_2\) space.

Remark 3.6. Every gbsb*-T\(_2\) space is gbsb*-T\(_1\).

Theorem 3.7. Every gbsb*-space is gbsb*-T\(_{1/2}\).

Proof: Let \((X, \tau)\) be a gbsb*-space and \(A\) be any gbsb*-closed set in \(X\). Then \(X\setminus A\) is gbsb*-open in \(X\). Since \(X\) is gbsb*-space, then \(X\setminus A\) is open in \(X\) and so \(A\) is closed. By Theorem 2.10, \(A\) is sb*-closed. Since shows that \(X\) is gbsb*-T\(_{1/2}\).

Generalized b*-strongly b*-D\(_k\) spaces

Definition 4.1. A subset \(A\) of a topological space \(X\) is called a gbsb*-difference set (briefly gbsb*-D-set) if there exists \(U, V\in \text{gbsb*O}(X)\) such that \(U\neq X\) and \(A=U\setminus V\).

Theorem 4.2. Every proper gbsb*-open set is a gbsb*-D-set.

Proof: Let \(A\) be any proper gbsb*-open subset of a topological space \(X\). Take \(U=A\) and \(V=\emptyset\). Then \(A=U\setminus V\) and \(U\neq X\). Hence \(A\) is gbsb*-D-set.

Remark 4.3. The converse of the above theorem need not be true which is shown in the following example.

Example 4.4. Let \(X=\{a,b,c,d\}\) with a topology \(\tau=\{\emptyset, \{a\}, \{a,b\}, \{a,b,c\}, X\}\). Then gbsb*-O\((X, \tau)\)=\(\{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, X\}\). Take \(U=\{a,b,d\}\) and \(V=\{a,b,c\}\). Then \(U\neq X\) and \(A=U\setminus V=\{a,b,d\}\setminus \{a,b,c\} \neq \{d\}\) is gbsb*-D-set but not a gbsb*-open set.

Definition 4.5. A topological space \((X, \tau)\) is said to be

(i) gbsb*-D\(_0\) if for any pair of distinct points \(x\) and \(y\) of \(X\) there exists a gbsb*-D-set of \(X\) containing \(x\) but not \(y\) or a gbsb*-D-set of \(X\) containing \(y\) but not \(x\).
Proposition 4.6. In a topological space $(X, \tau)$,
(i) if $(X, \tau)$ is gbsb*-T$_k$, then it is gbsb*-D$_k$, for $k = 0, 1, 2$.
(ii) if $(X, \tau)$ is gbsb*-D$_2$, then it is gbsb*-D$_{3-k}$, for $k = 1, 2$.

\textbf{Proof.}
(i) First we prove the result for $k=0$. Suppose $(X, \tau)$ is gbsb*-T$_0$. Then for each pair of distinct points $x$, $y$ in $X$, there exists a gbsb*-open set $U$ such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$. By Theorem 4.2, $U$ is gbsb*-D$_0$-set in $X$. Then we have for each pair of distinct points $x$, $y$ in $X$, there exists a gbsb*-D$_0$-set $U$ such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$. Hence $(X, \tau)$ is a gbsb*-D$_0$ space. Similarly we can prove that every gbsb*-T$_k$ space is gbsb*-D$_k$ space, for $k=1,2$.
(ii) Let $k=2$. Suppose $(X, \tau)$ is a gbsb*-D$_2$ space. Then for any pair of distinct points $x$ and $y$ of $X$, there exists disjoint gbsb*-D$_k$-sets $U$ and $V$ of $X$ containing $x$ and $y$ respectively. That is for any pair of distinct points $x$ and $y$ of $X$, there exists a gbsb*-D$_2$-set $U$ of $X$ containing $x$ but not $y$ and a gbsb*-D$_2$-set $V$ of $X$ containing $y$ but not $x$. Hence $(X, \tau)$ is a gbsb*-D$_2$ space. Similarly we can prove that every gbsb*-D$_1$ space is a gbsb*-D$_3$ space.

Theorem 4.7. A space $X$ is gbsb*-D$_0$ if and only if it is gbsb*-T$_0$.

\textbf{Proof.} Necessity: Suppose that $X$ is gbsb*-D$_0$. Then for each distinct pair $x$, $y$ in $X$, there is a gbsb*-D$_0$-set $G$ containing $x$ or $y$, say $x$ but not $y$. Since $G$ is gbsb*-D$_0$-set, then there are two gbsb*-open sets $U_1$ and $U_2$ such that $U_1 \neq X$ and $G = U_1 \cup U_2$. Since $x \notin G$ and $y \in G$, then $x \in U_1$. For $y \notin G$, we have two cases,
(a) $y \notin U_1$
(b) $y \in U_1$ and $y \notin U_2$.

In case (a), $x \notin U_1$ and $y \notin U_1$. In case (b), $y \in U_2$ and $x \notin U_2$. Thus in both cases we have for each pair of distinct points $x$ and $y$ in $X$, there exists a gbsb*-open set $U_1$ containing $x$ but not $y$ or a gbsb*-open set $U_2$ containing $x$ but not $y$. Hence $X$ is gbsb*-T$_0$.

Sufficiency: Suppose $(X, \tau)$ is gbsb*-T$_0$. Then by Theorem 4.6(i), $(X, \tau)$ is gbsb*-D$_0$.

Theorem 4.8. A space $X$ is gbsb*-D$_1$ if and only if it is gbsb*-D$_2$.

\textbf{Proof.} Necessity: Let $x$, $y$ in $X$, with $x \neq y$. Then there exist gbsb*-D$_1$-sets $G_1$, $G_2$ in $X$ such that $x \in G_1$, $y \notin G_1$, and $y \in G_2$, $x \notin G_2$. Since $G_1$ and $G_2$ are gbsb*-D$_1$-sets, then $G_1 = U_1 \cup U_2$ and $G_2 = U_3 \cup U_4$, where $U_1$, $U_2$, $U_3$, and $U_4$ are gbsb*-open sets in $X$. From $x \notin G_2$, it follows that either $x \notin U_3$ or $x \in U_3$, $x \in U_2$. We discuss the two cases separately.

(i) Suppose $x \notin U_3$. For $y \notin G_1$, we have two sub-cases:
(a) Suppose $y \notin U_1$. Since $x \notin U_1 \cup U_2$, it follows that $x \in U_1 \setminus (U_2 \cup U_3)$, and since $y \in U_3 \cup U_4$, we have $y \notin U_1 \setminus (U_2 \cup U_3)$. Since the union of gbsb*-open sets is gbsb*-open set, then $U_2 \cup U_3$ and $U_1 \cup U_4$ are gbsb*-open sets. Also $(U_1 \setminus (U_2 \cup U_3)) \cap (U_4 \setminus (U_1 \cup U_2)) = \emptyset$. Thus we have disjoint gbsb*-D$_1$-sets $U_1 \setminus (U_2 \cup U_3)$ and $U_4 \setminus (U_1 \cup U_2)$ containing $x$ and $y$ respectively.
(b) Suppose $y \notin U_2$. Since $x \notin U_1 \cup U_2$, we have $x \in U_3 \cup U_4$. Also $(U_1 \setminus U_2) \cap U_2 = \emptyset$. Thus we have disjoint gbsb*-D$_1$-sets $U_1 \setminus U_2$ and $U_2$ containing $x$ and $y$ respectively.

(ii) Suppose $x \in U_3$ and $x \in U_4$. We have $y \notin U_3 \cup U_4$. Hence $(U_3 \setminus U_4) \cap U_4 = \emptyset$. Thus we have disjoint gbsb*-D$_1$-sets $U_3$ and $U_4$ containing $x$ and $y$ respectively. Hence $X$ is gbsb*-D$_2$. 

Sufficiency: Suppose $X$ is $\text{gbsb}^*-\text{D}_2$. Then by Theorem 4.6(ii), $X$ is $\text{gbsb}^*-\text{D}_1$.

**Definition 4.9.** A point $x \in X$ which has only $X$ as the $\text{gbsb}^*$-neighbourhood is called a $\text{gbsb}^*$-neat point.

**Theorem 4.10.** For a $\text{gbsb}^*-\text{T}_0$ space $(X, \tau)$ the following are equivalent:

(i) $(X, \tau)$ is $\text{gbsb}^*-\text{D}_1$,

(ii) $(X, \tau)$ has no $\text{gbsb}^*$-neat point.

Proof. (i)⇒(ii). Since $(X, \tau)$ is $\text{gbsb}^*-\text{D}_1$, then each point $x$ of $X$ is contained in a $\text{gbsb}^*$-D-set $A = U \setminus V$ and thus in $U$. By definition $U \neq X$. This implies that $x$ is not a $\text{gbsb}^*$-neat point.

(ii)⇒(i) Suppose $(X, \tau)$ has no $\text{gbsb}^*$-neat point. Let $x$ and $y$ be distinct points in $X$. Since $X$ is $\text{gbsb}^*-\text{T}_0$, then there exists a $\text{gbsb}^*$-open set $U$ containing $x$ or $y$, say $x$. Since $y \notin U$, then $U \neq X$. By Theorem 4.2, $U$ is a $\text{gbsb}^*$-D-set. Since $X$ has no $\text{gbsb}^*$-neat point, then $y$ is not a $\text{gbsb}^*$-neat point. This means that there exists a $\text{gbsb}^*$-neighbourhood $V$ of $y$ such that $V \neq X$. Since $V$ is a $\text{gbsb}^*$-nbhd of $y$, there exists a $\text{gbsb}^*$-open set $G$ such that $y \in G \subseteq V$. Thus $y \in G \setminus U$ but not $x$. Also $G \setminus U$ is a $\text{gbsb}^*$-D-set. Hence $X$ is a $\text{gbsb}^*-\text{D}_1$ space.

**Corollary 4.11.** A $\text{gbsb}^*-\text{T}_0$ space $X$ is not $\text{gbsb}^*-\text{D}_1$ if and only if there is a unique $\text{gbsb}^*$-neat point in $X$.

Proof: Suppose $(X, \tau)$ be a $\text{gbsb}^*-\text{T}_0$ space. But $(X, \tau)$ is not a $\text{gbsb}^*-\text{D}_1$. Then by the above theorem $(X, \tau)$ has a $\text{gbsb}^*$-neat point. Now we have to prove the uniqueness. Suppose $x$ and $y$ are two different $\text{gbsb}^*$-neat points in $X$. Since $X$ is $\text{gbsb}^*-\text{T}_0$, at least one of $x$ and $y$, say $x$, has a $\text{gbsb}^*$-open set $U$ containing $x$ but not $y$. Then $U$ is a $\text{gbsb}^*$-nbhd of $x$ and $U \neq X$. Therefore $x$ is not a $\text{gbsb}^*$-neat point which contradicts $x$ is a $\text{gbsb}^*$-neat point. Hence $x = y$.

**References**